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Invariant solutions of the singular vortex in magnetohydrodynamics

Sergey V Golovin

Lavrentyev Institute of Hydrodynamics SB RAS, Novosibirsk 630090, Russia

E-mail: sergey@hydro.nsc.ru

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Abstract

We investigate properties of plasma flow, governed by an exact reduction of the ideal magnetohydrodynamics equations which was obtained in our recent article. The reduction is called a singular vortex. It gives a description of a three-dimensional nonstationary plasma flow in terms of solution of reduced system of PDEs with two independent variables. In this paper we investigate symmetry properties of the reduced system and construct its invariant solutions. Also we give a description of the physical features of plasma motion, governed by the singular vortex. Namely, we prove that in the observed class of solutions the particles trajectories as well as magnetic field lines are flat curves. The position and orientation of the plane containing the particle's trajectory depends on the initial position of the particle. We describe the shape of trajectories and magnetic field lines for the cases of stationary and self-similar plasma flow.

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1. Introduction

In our recent paper [1] we obtained an exact reduction of the ideal magnetohydrodynamics equations, which is called a singular vortex. It is a partially invariant solution of defect 1, constructed on the group $O(3)$ of rotations in the subspaces $\mathbb{R}^3(\mathbf{x})$ of coordinates, $\mathbb{R}^3(\mathbf{u})$ of velocity and $\mathbb{R}^3(\mathbf{H})$ of the magnetic field. The overdetermined system of PDEs, which defines the solution, was completed to involution. It was shown that the irreducible solution (i.e., solution with functional arbitrariness) is distinguished by the condition of the collinearity of the velocity, magnetic field and radius-vector of any fluid particle. The description of the singular vortex in MHD was split into two steps. In the first step, one has to investigate an invariant subsystem of PDEs with two independent variables. Its solutions describe the dependence of all the sought functions on time t and a radial coordinate r. In the second step, one has to determine the initial field of directions on the sphere $r = \text{const}$ from the finite (not differential) relation, which is observed on the solutions of an invariant subsystem. The field of directions allows one to describe the fluid motion as a whole.

In the present paper, we focus on the investigation of general properties of the fluid motions, governed by the singular vortex, and on the analysis of the invariant subsystem. It is known [2] that in the case of ideal gas dynamics (i.e., the magnetic field vanishes) the particles trajectories are flat curves. We prove that the same is also valid for MHD. Moreover, the magnetic field lines are also flat curves, belonging to the same plane as trajectories of the particles, which were initially located on the magnetic field line. We give the reduced equations for determination of the trajectories and magnetic field lines.

We also perform a symmetry analysis of the invariant system, describing the singular vortex and construct its invariant solutions. There are three nontrivial invariant solutions, which are described by the systems of ODEs. Namely, these are stationary, logarithmic and self-similar solutions. It is shown that in the stationary solution the velocity and magnetic field vectors are collinear. However, this is not valid in non-stationary case.

2. Preliminary information

The equations, governing the motions of ideal fluid with infinite conductivity are the following:

$$\begin{aligned}
&D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H}\times\mathrm{rot}\,\mathbf{H} = 0,\\
&Dp + A(p,\rho)\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{H} + \mathbf{H}\,\mathrm{div}\,\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{u} = 0,\\
&\mathrm{div}\,\mathbf{H} = 0, \qquad D = \partial_t + \mathbf{u}\cdot\nabla.
\end{aligned} \tag{2.1}$$

Here $\mathbf{u} = (u, v, w)$ is the velocity vector, p, ρ are the pressure and density, $\mathbf{H} = (H^1, H^2, H^3)$ is the magnetic field. All functions depend on time t and coordinates (x, y, z). Here and further $\partial_s \equiv \partial/\partial s$ is a partial derivative operator. The function $A(p,\rho)$ is defined by the state equation of the fluid, namely $A(p,\rho) = \rho c^2$, where c is a thermodynamic sound speed $c^2 = \partial p/\partial\rho$. In the polytropic gas $p = S\rho^\gamma$, where S is an entropy and γ is the adiabatic exponent, $A(p,\rho) = \gamma p$.

For given vectors $\mathbf{u}$ and $\mathbf{H}$, one can calculate the electric field as $\mathbf{E} = -\mathbf{u}\times\mathbf{H}$ and the electric current density $\mathbf{j} = \mathrm{rot}\,\mathbf{H}$.

For convenience we introduce the spherical coordinate system

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta. \tag{2.2}$$

The decomposition of any vector field $\mathbf{u} = (u, v, w)$ on the spherical reference frame has the form

$$\begin{aligned}
&u_r = u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta,\\
&u_\theta = u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta,\\
&u_\varphi = -u\sin\varphi + v\cos\varphi.
\end{aligned} \tag{2.3}$$

Vectors $\mathbf{u}$ and $\mathbf{H}$ are decomposed by spherical frame according to (2.3). The following individual notations of components of velocity and magnetic field vectors are introduced (see figure 1)

$$\begin{aligned}
&v_r = U, \qquad v_\theta = M\cos\Omega, \qquad v_\varphi = M\sin\Omega;\\
&H_r = H, \qquad H_\theta = N\cos\Sigma, \qquad H_\varphi = N\sin\Sigma.
\end{aligned} \tag{2.4}$$

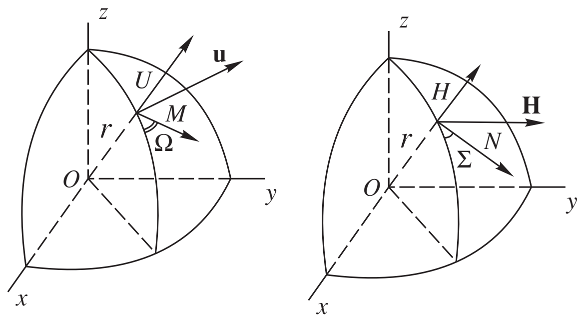

Figure 1. The decomposition of the velocity and magnetic field vectors.

Here U and H are the radial components of $\mathbf{u}$ and $\mathbf{H}$. Functions M and N denote respectively absolute values of the components of the velocity and magnetic field vectors tangential to spheres $r = \text{const}$. Functions Ω and Σ are the angles between tangential components of $\mathbf{u}$ and $\mathbf{H}$ and the meridional direction.

The exact solution of interest is distinguished by the requirement for vector fields $\mathbf{u}$ and $\mathbf{H}$ to be partially invariant with respect to the group $O(3)$ of synchronous rotations in the spaces $\mathbb{R}^3(\mathbf{x})$, $\mathbb{R}^3(\mathbf{u})$ and $\mathbb{R}^3(\mathbf{H})$. One can check that functions U, M, H and N as well as thermodynamical values p and ρ are invariants of the group $O(3)$. Besides, the difference $\Omega - \Sigma$ is also an invariant function. However, angles Ω and Σ themselves are not invariant under the action of $O(3)$. According to the algorithms of group analysis of differential equations, all invariant functions are set to depend on invariant independent variables t and r only. The non-invariant functions Ω and Σ are supposed to depend on all the independent variables t, r, θ, φ, in that the difference $\Omega - \Sigma$ is function of invariant variables t and r only:

$$\Omega = \omega(t, r, \theta, \varphi), \qquad \Sigma = \sigma(t, r) + \omega(t, r, \theta, \varphi).$$

Substitution of this representation into MHD equations (2.1) provides a system Π of nine equations for invariant functions U, H, N, M, ρ, p and a non-invariant function ω. This system should be observed as an overdetermined system of the first-order PDEs for function $\omega(t, r, \theta, \varphi)$ under assumption that all the invariant functions are known. The compatibility conditions for this system give the equations for invariant functions. In order to omit known solutions, we observe only the case when the function ω is determined with a functional arbitrariness. Function ω has only constant arbitrariness if it is possible to express all the first-order derivatives of ω from the system Π. To impose a ban on this situation, we calculate a matrix of coefficients of derivatives of function ω and claim it to be of rank 3 or less. In [1], it is shown that the claim is satisfied only in case when the velocity vector $\mathbf{u}$ and magnetic field vector $\mathbf{H}$ in each point are coplanar to the radius-vector of the point. Thus, the most general representation of the solution of the singular vortex type for ideal MHD is

$$\begin{aligned} &U = U(t, r), \qquad M = M(t, r), \qquad H = H(t, r), \qquad N = N(t, r), \\ &\Sigma = \Omega = \omega(t, r, \theta, \varphi), \qquad p = p(t, r), \qquad \rho = \rho(t, r). \end{aligned} \tag{2.5}$$

The analysis of equations (2.1) on solution (2.5) was performed in [1]. We use the notations

$$M_1 = r^{-1}M, \qquad H_1 = r^2 H, \qquad N_1 = rN, \qquad H_1 = H_0(\cos\tau)^{-1}. \tag{2.6}$$

Here H_0 is an arbitrary constant. In the polytropic case $A(p, \rho) = \gamma p$ the constant $H_0 \neq 0$ could be set $H_0 = 1$ by the dilatation transformation $2\rho\partial_\rho + 2p\partial_r + H\partial_H + N\partial_N$, which is admitted by equations (2.1). However, we will preserve the constant H_0 since it allows one to keep the case of zero magnetic field $\mathbf{H} = 0$ as a limit $H_0 \to 0$, $N \to 0$.

The system of equations for invariant functions can be written in the following form:

$$
\begin{aligned}
&D_0 M_1 + \frac{2}{r} U M_1 - \frac{H_0}{r^4 \rho \cos\tau} N_{1r} = 0, \\
&D_0 N_1 + N_1 U_r - \frac{H_0}{\cos\tau} M_{1r} - M_1 N_1 \tan\tau = 0, \\
&D_0 p + A(p,\rho)\left(U_r + \frac{2}{r} U - M_1 \tan\tau\right) = 0, \\
&D_0 U + \frac{1}{\rho} p_r + \frac{N_1 N_{1r}}{r^2 \rho} - r M_1^2 = 0, \qquad H_0 \tau_r = N_1 \cos\tau, \\
&D_0 \rho + \rho\left(U_r + \frac{2}{r} U - M_1 \tan\tau\right) = 0, \qquad D_0 \tau = M_1, \\
&D_0 = \partial_t + U \partial_r .
\end{aligned}
\tag{2.7}
$$

Equations (2.7) form an overdetermined system of seven equations for the six sought functions. However, this system is in involution, since the compatibility condition of the last two equations for τ is the second equation for N_1 in system (2.7). The initial data for the well-posed Cauchy problem for system (2.7) are

$$
\begin{aligned}
&M_1(t_0, r) = m(r), \qquad N_1(t_0, r) = n(r), \qquad p(t_0, r) = p_0(r), \\
&U(t_0, r) = u_0(r), \qquad \rho(t_0, r) = \rho_0(r), \qquad \tau(t_0, r_0) = \tau_0,
\end{aligned}
\tag{2.8}
$$

where m, n, p_0, u_0, ρ_0 are the arbitrary functions of r; τ_0 is a constant. Therefore, the function $\tau(t, r)$ is defined by its value on a fixed sphere $r = r_0$ at initial time $t = t_0$.

The non-invariant function ω is restricted by the following overdetermined system:

$$
\begin{aligned}
&N_1 \omega_t + (N_1 U - H_1 M_1)\omega_r = 0, \\
&H_1 \cos\omega\, \omega_r + N_1 \omega_\theta - N_1 \tan\tau \sin\omega = 0, \\
&\sin\theta \sin\omega\, \omega_\theta - \cos\omega\, \omega_\varphi - \tan\tau \sin\theta - \cos\theta \cos\omega = 0.
\end{aligned}
\tag{2.9}
$$

System (2.9) is in involution on the solutions of the invariant system (2.7). Equations (2.9) could be completely integrated. The general solution of (2.9) is represented in an implicit form as

$$
F(\eta, \zeta) = 0, \tag{2.10}
$$

where F is an arbitrary smooth function of two arguments and

$$
\begin{aligned}
&\eta = \sin\theta \cos\omega \cos\tau - \cos\theta \sin\tau, \\
&\zeta = \varphi + \arctan \frac{\sin\omega \cos\tau}{\cos\theta \cos\omega \cos\tau + \sin\theta \sin\tau}.
\end{aligned}
\tag{2.11}
$$

Equation (2.10) implicitly defines function $\omega = \omega(t, r, \theta, \varphi)$. Note that the function ω depends on the variables t and r only by means of function $\tau(t, r)$. Further, we will need the equations, describing the evolution of the function ω along the particle's trajectory and along the magnetic force line. The operators D and $r^{-2}\widetilde{D}$ of the differentiation along the trajectory and magnetic force line correspondingly in the spherical coordinate system (2.2) have the following form:

$$
\begin{aligned}
&D = \partial_t + U\partial_r + M_1 \cos\omega \partial_\theta + \frac{M_1 \sin\omega}{\sin\theta} \partial_\varphi, \\
&\widetilde{D} = H_1 \partial_r + N_1 \cos\omega \partial_\theta + \frac{N_1 \sin\omega}{\sin\theta} \partial_\varphi.
\end{aligned}
$$

Linear combinations of the equations of system (2.9) give

$$\sin\theta\, D\omega = -M_1 \cos\theta \sin\omega, \tag{2.12}$$

$$\sin\theta\, \widetilde{D}\omega = -N_1 \cos\theta \sin\omega. \tag{2.13}$$

3. Particle's trajectory and magnetic force line

Let us calculate the vector which is orthogonal to the plane containing the radius-vector and velocity vector of a particle at an initial moment of time

$$\mathbf{k} = \mathbf{x} \times \mathbf{u} = r^2 M_1 \mathbf{n},$$

where

$$\mathbf{n} = \cos\omega(\sin\varphi, -\cos\varphi, 0) + \sin\omega(\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta). \tag{3.1}$$

Here vector $\mathbf{n}$ is decomposed on the basis of the Cartesian coordinate system.

Theorem 1. *Vector* $\mathbf{n}$ *is constant along the particle's trajectory and along the magnetic force line, which passes through the particle at some moment of time.*

The statement of theorem 1 is verified by direct computation. Actually, according to expressions (2.12) and (2.13) we have

$$D\mathbf{n} = \widetilde{D}\mathbf{n} = 0.$$

Theorem 1 means that the particles' trajectories and magnetic force lines in the singular vortex (2.5) are flat curves. The plane of the particle's motion is determined by the initial position and velocity vector of the particle.

In order to describe the trajectory of particle, let us introduce a polar coordinate system (r, ψ) in the plane of its motion. Here r is the distance to the origin of the spherical system (2.2) and ψ is a polar angle which is counted off from the particle's initial position. In this coordinate system, the particle's motion is described by the solutions of the following Cauchy problem:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U(t,r), \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t} = M_1(t,r), \qquad r(t_0) = r_0, \qquad \psi(t_0) = 0, \tag{3.2}$$

where U and M_1 are the solutions of the invariant system (2.7). The magnetic force line, which passes through the chosen particle, is defined by the equations

$$\frac{\mathrm{d}r}{H_1} = \frac{\mathrm{d}\psi}{N_1}, \qquad r|_{\psi=\psi(t)} = r(t). \tag{3.3}$$

Here $r(t)$ and $\psi(t)$ are the functions, obtained from a solution of the Cauchy problem (3.2). The time t in equation (3.3) is observed as a parameter. At all moments of time the magnetic force line (3.3) belongs to the same plane as the trajectory (3.2). Note that trajectories of all particles which pass through the magnetic force line belong to the same plane as the magnetic line. In fact, the magnetic force line is a liquid curve, i.e. it consists of the same particles at all moments of time.

Let us observe the equations for the function τ in the system (2.7):

$$\tau_t + U\tau_r = M_1, \tag{3.4}$$

$$H_1\tau_r = N_1. \tag{3.5}$$

The characteristics equations for equation (3.4) coincide with the trajectories equations (3.2). Equation (3.5) coincides with the magnetic line equation (3.3). Thus, the polar angle ψ of a particle is determined by the function τ. Similarly, the polar angle of the points of the magnetic force lines at the time t is determined by the function τ.

For given invariant functions satisfying the Cauchy problem (2.7), (2.8) the particle's trajectory and the magnetic force lines can be constructed by the following procedure. The dependence $r = r(t)$ is determined from the Cauchy problem

$$\frac{\mathrm{d}r}{\mathrm{d}t} = U(t, r), r(t_0) = r_*. \tag{3.6}$$

The trajectory of the particle, which starts at $t = t_0$ from the sphere $r = r_*$, is parametrically defined in the polar coordinates (r, ψ) by the equations

$$r = r(t), \qquad \psi = \tau(t, r(t)) - \tau(t_0, r_*). \tag{3.7}$$

The equation of the magnetic force line is

$$\psi = \tau(t, r) - \tau(t_0, r_*). \tag{3.8}$$

In contrast to the trajectory equation (3.7), the time t should be observed as a parameter in equation (3.8). As will be shown later, in the stationary case the velocity vector $\mathbf{u}$ is collinear with the magnetic field vector $\mathbf{H}$. In this case, the streamlines coincide with the magnetic force lines and both are defined by the formula $\psi = \tau(r) - \tau(r_*)$.

From the above, it follows that the shapes of the trajectories of the particles, which start from the same sphere, are identical. The location of each trajectory in the space is defined by the initial position and initial direction of the velocity vector of each particle. Therefore, the whole motion is defined by two factors:

- the solution of the invariant system (2.7), which determines the shape of trajectories;
- the function ω, which is defined by an implicit equation (2.10) and assigns the location of the trajectories in the 3D space.

Thus, the plasma motion, governed by the observed solution, is nontrivial and sufficiently three dimensional.

Theorem 2. *The trajectories of the particles and magnetic force lines are flat curves. The particles, which start from the same sphere, move on identical curves in the planes of their motion. The location and orientation of the plane depend on the particle's initial position and are defined by the function ω, which is determined by an implicit equation (2.10).*

4. The symmetry properties of the invariant system

The direct calculations allow us to prove the following statement.

Theorem 3. *In the case of polytropic gas with the state equation $p = S\rho^\gamma$, $A(p, \rho) = \gamma p$ the invariant system (2.7) admits a three-dimensional Lie group G_3 of transformations, consisting of a time-shift and two dilatations. The corresponding Lie algebra L_3 is generated by the following infinitesimal operators:*

$$\begin{aligned} X_1 &= \partial_t, \\ X_2 &= t\partial_t - U\partial_U - M_1\partial_{M_1} + 2\rho\partial_\rho, \\ X_3 &= r\partial_r + U\partial_U - N_1\partial_{N_1} - 4p\partial_p - 6\rho\partial_\rho. \end{aligned} \tag{4.1}$$

Besides, the following discrete transformations are admitted:

$$\varepsilon_1 : t \to -t, U \to -U, M_1 \to -M_1,$$
$$\varepsilon_2 : r \to -r, U \to -U, N_1 \to -N_1,$$
$$\varepsilon_3 : \tau \to -\tau, M_1 \to -M_1, N_1 \to -N_1.$$

The discrete transformation ε_2 looks non-physical since $r > 0$ by its definition. However, the transformation $r \to -r$ is equivalent to the discrete shift $\varphi \to \varphi + \pi, \theta \to \pi - \theta$; hence it is also physically warrantable.

The optimal system of subalgebras [3, 4] of the Lie algebra L_3 is easily constructed in the form

$$\dim = 1 : \{X_1\}, \{X_1 + X_3\}, \{X_2 + \alpha X_3\}, \{X_3\},$$
$$\dim = 2 : \{X_1, X_3\}, \{X_2, X_3\}, \{X_1, X_2 + \alpha X_3\},$$
$$\dim = 3 : \{X_1, X_2, X_3\}.$$

As usual $\{X_1, \ldots, X_i\}$ denotes the Lie subalgebra with basic elements $X_1, \ldots, X_i$. The one-dimensional representatives of the optimal system produce the reductions of system (2.7) to the ordinary differential equations. There are four such reductions. Further, we will observe them one after another. Note that symmetry reductions for the singular vortex in gas dynamics were observed in [5, 6].

5. The stationary solution

The subalgebra $\{X_1\}$ generates a stationary solution of system (2.7) since its invariants are all the sought functions and an independent variable r. All the invariant functions are supposed to depend only on r. Equations (2.7) are reduced to the following system of ordinary differential equations:

$$UM_1' + \frac{2}{r}UM_1 - \frac{H_0 N_1'}{r^4 \rho \cos\tau} = 0, \tag{5.1}$$

$$UN_1' + N_1 U' - \frac{H_0 M_1'}{\cos\tau} - M_1 N_1 \tan\tau = 0, \tag{5.2}$$

$$Up' + \gamma p \left(U' + \frac{2}{r}U - M_1 \tan\tau \right) = 0, \tag{5.3}$$

$$UU' + \frac{1}{\rho}p' + \frac{N_1 N_1'}{r^2 \rho} - rM_1^2 = 0, \tag{5.4}$$

$$U\rho' + \rho \left(U' + \frac{2}{r}U - M_1 \tan\tau \right) = 0, \tag{5.5}$$

$$H_0 \tau' = N_1 \cos\tau, \qquad U\tau' = M_1. \tag{5.6}$$

Here prime denotes the derivative with respect to r. Compatibility condition for equations (5.6) gives

$$UN_1 \cos\tau = M_1 H_0. \tag{5.7}$$

According to (2.6) it is equivalent to $UN = MH$. This means that under the considered assumptions the vectors **u** and **H** are collinear. Equation (5.2) is satisfied by virtue of (5.7). From (5.3) and (5.5), we obtain the entropy conservation

$$S = S_0 = \text{const}. \tag{5.8}$$

Transformation of continuity equation (5.5), taking into account equation (5.6), gives the discharge rate integral

$$r^2\rho U\cos\tau = n^{-1}, \qquad n = \text{const}. \tag{5.9}$$

From the latter and (5.7), it follows that $N_1 = nH_0r^2\rho M_1$. Without loss of generality we assume that $n = 1$, which could be achieved by the discrete transformation ε_1 and dilatation X_2.

Equation (5.1) with the aid of relation (5.9) can be integrated as

$$r^2M_1 = H_0N_1 + m, \qquad m = \text{const}. \tag{5.10}$$

In the case of $m \neq 0$, we use an involution ε_3 and dilatation $3X_2 + X_3$ to make $m = 1$. One can express M_1 and N_1 in terms of ρ,

$$M_1 = \frac{1}{r^2\left(1 - H_0^2\rho\right)}, \qquad N_1 = \frac{H_0\rho}{1 - H_0^2\rho}.$$

Substitution of these relations into (5.4) allows us to integrate it as

$$U^2 + \frac{2\gamma S_0}{\gamma - 1}\rho^{\gamma-1} + \frac{1}{r^2\left(1 - H_0^2\rho\right)^2} = b^2, \qquad b = \text{const}. \tag{5.11}$$

Relation (5.11) is the Bernoulli integral. The only equation left is (5.6). Let us introduce an auxiliary function

$$\tau = 2\arctan\left[\tanh\left(\frac{1}{2}\sigma\right)\right]. \tag{5.12}$$

Expressions for all the sought functions in terms of σ are

$$M_1 = \frac{1 + H_0^2\sigma'}{r^2}, \qquad N_1 = H_0\sigma', \qquad \rho = \frac{\sigma'}{1 + H_0^2\sigma'}, \qquad U = \frac{\left(1 + H_0^2\sigma'\right)}{\sigma'}\frac{\cosh\sigma}{r^2}. \tag{5.13}$$

Substitution of the obtained representations into Bernoulli integral (5.11) gives an equation for σ determination:

$$\left(\frac{1 + H_0^2\sigma'}{\sigma'}\right)^2\frac{\cosh^2\sigma}{r^4} + \frac{2\gamma S_0}{\gamma - 1}\left(\frac{\sigma'}{1 + H_0^2\sigma'}\right)^{\gamma-1} + \frac{\left(1 + H_0^2\sigma'\right)^2}{r^2} = b^2. \tag{5.14}$$

Equation (5.14) is not solved with respect to derivative σ'. Nevertheless, it can be integrated in parametrical form since it is solvable with respect to either σ or r.

Let us investigate separately the case $m = 0$ in integral (5.10). Equations (5.7), (5.9) and (5.10) give

$$r^2U\cos\tau = H_0^2, \qquad H_0N_1 = r^2M_1, \qquad \rho = H_0^{-2}. \tag{5.15}$$

Substitution of the expressions for ρ and N_1 into equation (5.4) allows the following integration:

$$U^2 + r^2M_1^2 = b^2, \qquad b = \text{const}. \tag{5.16}$$

As before, we use the dilatation $3X_2 + X_3$ to make $b = H_0^2$. We introduce an auxiliary function σ by formula (5.12). The expressions for all the sought functions in terms of σ are

$$M_1 = \frac{H_0^2\sigma'}{r^2}, \qquad N_1 = H_0\sigma', \qquad U = \frac{H_0^2\cosh\sigma}{r^2}, \qquad \rho = \frac{1}{H_0^2}. \tag{5.17}$$

Substitution of expressions (5.17) into Bernoulli integral (5.16) gives

$$\sigma'^2 = r^2 - r^{-2}\cosh^2\sigma. \tag{5.18}$$

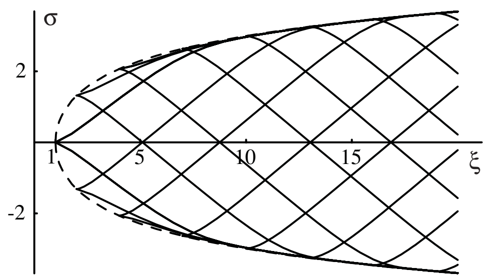

Figure 2. Integral curves of equation (5.19).

Let us investigate equation (5.18) in detail. It is convenient to introduce the new variable $\xi = r^2$. The equation is transformed to the following:

$$4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\xi}\right)^2 = 1 - \frac{\cosh^2\sigma}{\xi^2}. \tag{5.19}$$

Real-valued solutions of this equations exist only in the definition domain $\xi > \cosh\sigma$. The curve $\xi = \cosh\sigma$, which is the boundary of the definition domain, is called the discriminant curve. The origin $r = 0$ lies outside the definition domain of equation (5.19), therefore, the observed solution of the MHD equations cannot be prolonged up to the origin $r = 0$. The solution exists outside some sphere $r = r_* = \sqrt{\xi_*}$, where ξ_* is a minimal possible value of ξ on the chosen integral curve of equation (5.19). The sphere $r = r_*$ should be observed as a source or a drain of fluid and of the magnetic field since the velocity and magnetic field vectors have some nonzero values on that sphere. According to formulae (5.17), vector fields $\mathbf{u}$ and $\mathbf{H}$ have radial directions on the sphere $r = r_*$. Let us show that the limiting direction of $\mathbf{u}$ and $\mathbf{H}$ at $r \to \infty$ is also a radial one.

Theorem 4. *Equation (5.19) has the following properties.*

(a) At any point of the definition domain there are two integral curves: one is monotonically increasing and the other one is monotonically decreasing.
(b) Integral curves have a horizontal tangent at the points of the discriminant curve.
(c) The limiting behaviour at $\xi \to \infty$ of the integral curves is $\mathrm{d}\sigma/\mathrm{d}\xi \to 0$, $\xi^{-1}\cosh\sigma \to 1$.

Proof of the theorem is given in the appendix. The picture of integral curves of equation (5.19) is shown in figure 2. The dashed line is a discriminant curve.

According to formulae (5.17) and statement (c) of theorem 4, the limiting at $r \to \infty$ velocity and magnetic vector fields are radial with the following asymptotic:

$$U \sim H_0^2, \qquad M \sim \frac{2H_0^2}{r^2}, \qquad H \sim H_0, \qquad N \sim \frac{2H_0}{r^2}.$$

It gives a radial recession of the particles and a radial magnetic field at the infinity.

Note that at the points of discriminant curve, the derivatives of functions M and N become infinite, though the functions themselves vanish. Therefore, the physically reasonable solution starts at some distance from the discriminant curve.

The streamlines of the flow, which coincide with the magnetic force lines, are calculated according to the algorithm given in section 3. The typical magnetic force line is shown in figure 3.

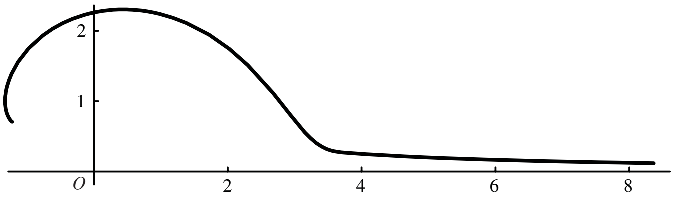

Figure 3. The typical magnetic field line in the stationary solution.

6. The self-similar reduction

The governing subalgebra is $\{X_2 + \alpha X_3\}$. Here we have the following representation of the invariant solution

$$U = t^{\alpha-1}(v(\lambda) + \alpha\lambda), \qquad M_1 = m(\lambda)t^{-1}, \qquad N_1 = n(\lambda)t^{-\alpha}, \qquad \tau = \tau(\lambda),$$
$$p = P(\lambda)t^{-4\alpha}, \qquad \rho = R(\lambda)t^{2-6\alpha}, \qquad S = s(\lambda)t^{6\alpha\gamma-2\gamma-4\alpha}, \qquad \lambda = rt^{-\alpha}. \tag{6.1}$$

Constant α can take on any real values. For the determinacy, we suppose $t > 0$. Equations for the determination of the invariant functions are

$$vm' + m\left(\frac{2v}{\lambda} + 2\alpha - 1\right) - \frac{H_0 n'}{\lambda^4 R\cos\tau} = 0, \tag{6.2}$$

$$vn' + nv' - \frac{H_0 m'}{\cos\tau} - mn\tan\tau = 0, \tag{6.3}$$

$$vP' + \gamma P\left(v' + \frac{2v}{\lambda} - m\tan\tau\right) = \alpha(4 - 3\gamma)P, \tag{6.4}$$

$$vv' + \frac{1}{R}P' + \frac{nn'}{\lambda^2 R} - \lambda m^2 = \alpha(1-\alpha)\lambda + (1-2\alpha)v, \tag{6.5}$$

$$vR' + R\left(v' + \frac{2v}{\lambda} - m\tan\tau\right) = (3\alpha - 2)R, \tag{6.6}$$

$$H_0\tau' = n\cos\tau, \qquad v\tau' = m. \tag{6.7}$$

Here the prime denotes the derivative with respect to λ. The entropy equation is the following:

$$vs' = (2\gamma + 4\alpha - 6\alpha\gamma)s. \tag{6.8}$$

The compatibility condition of equations (6.7) is

$$H_0 m = nv\cos\tau. \tag{6.9}$$

Equation (6.3) is satisfied by virtue of equation (6.9). Let us observe equation (3.6). In the invariant variables (6.1), the equation turns to

$$t\frac{\mathrm{d}\lambda}{\mathrm{d}t} = v(\lambda). \tag{6.10}$$

Equation (6.10) could be observed as a definition of a new independent variable t, while λ becomes a sought function, satisfying equation (6.10). The derivative transforms as

$$v\frac{\mathrm{d}}{\mathrm{d}\lambda} = t\frac{\mathrm{d}}{\mathrm{d}t}.$$

With this variable the continuity equation (6.6) is integrated as

$$Rv\lambda^2\cos\tau = Ct^{3\alpha-2}, \qquad C = \text{const}. \tag{6.11}$$

From (6.8) we have

$$s = S_0 t^{2\gamma+4\alpha-6\alpha\gamma}, \qquad S_0 = \text{const.} \tag{6.12}$$

The further integration is possible for the special value of the similarity exponent $\alpha = 1$. In this special case, one can obtain an integral of equation (6.2) in the form

$$C\lambda^2 mt = H_0 n + C_1, \qquad C_1 = \text{const.} \tag{6.13}$$

Let us now apply transformations admitted by the system (6.2)–(6.7) to simplify the obtained integrals. First we apply the time and space dilatation $t \to t/C$ to make $C = 1$ in integral (6.11).

(a) Let $C_1 \neq 0$. We use another admissible dilatation $\lambda \to a\lambda$, $v \to av$, $R \to a^{-6}R$, $n \to a^{-1}n$, $t \to t/a^3$ with $a = C_1^{-1}$ to make $C_1 = 1$. These transformations reflect on the constant S_0, which changes its value.

One could express functions m and n from integrals (6.11) and (6.13) as

$$m = \frac{t}{\lambda^2\left(t^2 - H_0^2 R\right)}, \qquad n = \frac{H_0 R}{t^2 - H_0^2 R}. \tag{6.14}$$

Substitution of relations (6.14) into (6.5) allows us to integrate the latter equation in the case of special adiabatic exponent $\gamma = 2$:

$$v^2 + m^2\lambda^2 + 4S_0 R t^{-4} = b^2 t^{-2}, \qquad b = \text{const.} \tag{6.15}$$

Expression (6.15) has the form of a Bernoulli integral.

There are two equations left. The first is one of the equations (6.7). The second one is equation (6.10) for the function $\lambda(t)$. Let us express all the sought functions in terms of derivatives of τ. In the case $C_1 = 1$, the expressions have the form

$$v = \frac{H_0^2 t\tau_t}{(\lambda^2 t^2 \tau_t - 1)\cos\tau}, \qquad R = \frac{t^2\lambda^2\tau_t - 1}{H_0^2\lambda^2\tau_t}, \qquad n = \frac{\lambda^2 t^2\tau_t - 1}{H_0}, \qquad m = t\tau_t. \tag{6.16}$$

Substitution into the Bernoulli integral (6.15) gives an implicit (not solved with respect to derivative τ_t) equation for function $\tau(t)$. The second equation is (6.10) for function $\lambda(t)$.

(b) Let $C_1 = 0$. From equations (6.9), (6.11) and (6.13), it follows that

$$R = t^2 H_0^{-2}, \qquad t v\lambda^2\cos\tau = H_0^2. \tag{6.17}$$

Integration of equation (6.5) gives Bernoulli integral for arbitrary γ

$$v^2 + m^2\lambda^2 + 2S_0 t^{2(1-\gamma)} = b^2 t^{-2}, \qquad b = \text{const.} \tag{6.18}$$

The expressions of the sought functions in terms of τ are

$$v = \frac{H_0^2}{\lambda^2 t\cos\tau}, \qquad R = \frac{t^2}{H_0^2}, \qquad n = \frac{\lambda^2 t^2\tau_t}{H_0}, \qquad m = t\tau_t. \tag{6.19}$$

From (6.7) and (6.10) we have the following system:

$$\tau_t = \pm\frac{1}{t^2\lambda}\sqrt{b^2 - 2S_0 t^{2(2-\gamma)} - \left(\frac{H_0^2}{\lambda^2\cos\tau}\right)^2}, \tag{6.20}$$

$$\lambda_t = \frac{H_0^2}{\lambda^2 t^2\cos\tau}. \tag{6.21}$$

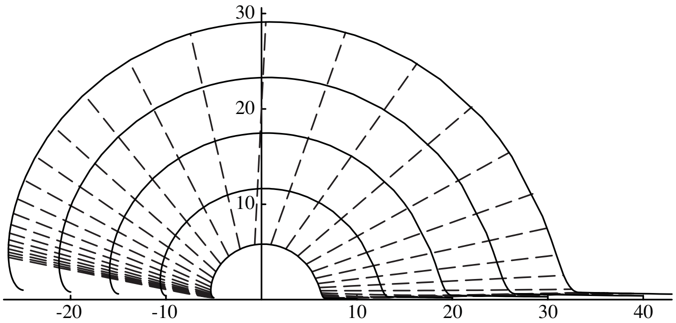

Figure 4. Magnetic force lines (continuous lines) and particles' trajectories (dashed lines) at the self-similar solution.

System (6.20) for $\gamma = 2$ is reduced to the following equation,

$$\frac{d\tau}{d\lambda} = \pm\sqrt{A^2\lambda^2\cos^2\tau - \lambda^{-2}}, \tag{6.22}$$

where $A^2 = H_0^{-4}(b^2 - 2S_0)$. One can make $A = 1$ by the suitable dilatation transformation. The introduction of the new sought function σ by formulae (5.12) and a new variable $\xi = \lambda^2$ reduces equation (6.22) to equation (5.19).

Equation (6.22) defines the dependence $\tau = \tau(\lambda)$. It serves for the description of the magnetic force lines. Actually, according to equation (3.8) the magnetic force line at the time $t = 1$ is described by the formula

$$\psi = \tau(\lambda) - \tau(\lambda_0).$$

Here ψ is a polar angle of a point on the magnetic line in a plane. For the subsequent time moments, the magnetic line enlarges homothetically. The typical magnetic force line coincides with the curve in figure 3. For each integral curve of equation (6.22), there exists a minimal value $\lambda = \lambda_*$ in the definition domain of function $\tau(\lambda)$. Thus, the solution is defined not in the whole space, but outside some sphere with an increasing radius $r = \lambda_* t$.

The formulae for particles trajectories are determined from equation (6.21), where the dependence $\tau(\lambda)$ follows from (6.22). Integration of (6.21) with initial data $\lambda(t_0) = \lambda_0$ gives

$$t = t_0\left(1 - t_0\int_{\lambda_0}^{\lambda}\sigma^2\cos\tau(\sigma)\,d\sigma\right)^{-1}. \tag{6.23}$$

Equation (6.23) shows that there exists a finite value $\lambda = \lambda_1$ such that $t \to \infty$. Hence along the trajectory for $t_0 \leqslant t \leqslant \infty$ the variable λ takes values $\lambda_0 \leqslant \lambda \leqslant \lambda_1$.

The typical magnetic field lines (continuous lines) for the time moments form $t = 1$ to $t = 5$ with the particles trajectories (dashed lines) are shown in figure 4. The smallest magnetic line corresponds to the time $t = 1$, the largest one is given for the time $t = 5$. The particles' trajectories start at the time $t = 1$ from the magnetic field line for λ varying from 1 to 1.4 with a step of 0.01.

7. The logarithmic reduction

The governing subalgebra is $\{X_1 + X_3\}$. Calculation of the invariants allows us to write the representation of the solution in the form:

$$U = r(v(\lambda)+1), \qquad M_1 = m(\lambda), \qquad N_1 = \frac{n(\lambda)}{r}, \qquad \tau = \tau(\lambda),$$
$$p = \frac{P(\lambda)}{r^4}, \qquad \rho = \frac{R(\lambda)}{r^6}, \qquad S = s(\lambda) r^{6\gamma-4}, \qquad \lambda = t - \ln r. \tag{7.1}$$

Substitution of the ansatz (7.1) into equations (2.7) gives

$$vm' - \frac{H_0(n'+n)}{R\cos\tau} - 2m(v+1) = 0, \tag{7.2}$$

$$vn' + nv' - \frac{H_0 m'}{\cos\tau} + mn\tan\tau = 0, \tag{7.3}$$

$$vP' + \gamma P(v' - 3(v+1) + m\tan\tau) = -4P(v+1), \tag{7.4}$$

$$vv' + \frac{P'+4P}{R} + \frac{n(n'+n)}{R} = (v+1)^2 - m^2, \tag{7.5}$$

$$vR' + R(v' + 3(v+1) + m\tan\tau) = 0, \tag{7.6}$$

$$H_0\tau' = -n\cos\tau, \qquad v\tau' = -m. \tag{7.7}$$

The equation for the entropy is

$$vs' = (6\gamma - 4)s(v+1). \tag{7.8}$$

The same compatibility condition for equations (7.7)

$$H_0 m = nv\cos\tau \tag{7.9}$$

serves as an integral of equation (7.3). As in a self-similar solution, we observe equation (3.6) for the particle's trajectory. In the invariant variables (7.1), it looks as follows:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -v(\lambda) \quad \Rightarrow \quad v\frac{\mathrm{d}}{\mathrm{d}\lambda} = -\frac{\mathrm{d}}{\mathrm{d}t}. \tag{7.10}$$

The introduction of a new independent variable t allows us to integrate the continuity equation (7.6) in the form,

$$RV\cos\tau\,\mathrm{e}^{3(\lambda-t)} = C, \qquad C = \text{const}. \tag{7.11}$$

The entropy equation (7.8) is integrated as

$$s = s_0\,\mathrm{e}^{(6\gamma-4)(\lambda-t)}. \tag{7.12}$$

Subsequent integration of the observed equations should be performed numerically.

8. The homogenous reduction

The governing subalgebra is $\{X_3\}$. The representation of the $\{X_3\}$-invariant solution has the form

$$U = ru(t), \qquad M_1 = m(t), \qquad N_1 = r^{-1}n(t),$$
$$\tau = \tau(t), \qquad \rho = r^{-6}R(t), \qquad p = r^{-4}P(t). \tag{8.1}$$

Substitution of representation (8.1) into the last two equations for function τ in system (2.1) gives the following:

$$n\cos\tau = 0, \qquad \tau' = m. \tag{8.2}$$

Here the prime denotes time derivative. From the first equation of (8.2) under assumption $\tau \neq \pi/2$ it follows that $n \equiv 0$. Thus, the observed case defines plasma motion with a purely radial magnetic field. Such motions were investigated in [1]. In particular, it was shown that the solution is described by one third-order ODE. We will not continue the investigation of homogeneous reduction of the singular vortex here, referring the interested reader to the mentioned article.

9. Conclusion

We investigated properties of plasma flow, which is governed by the solution of the singular vortex type. The following features of the solution are shown.

- The velocity vector, the magnetic field vector and the radius vector of any particle are coplanar.
- The particles trajectories and the magnetic field lines are flat curves. The position and orientation of the plane, which contain the trajectory curve, depend on the initial position of the particle.
- All particles, belonging to the same sphere $r = \text{const}$ at some moment of time $t = t_0$, have the same trajectory curve. However, the location of each trajectory in the 3D space is determined by the initial distribution of the velocity field on the sphere.
- In the stationary singular vortex, the velocity and magnetic vector fields are collinear, hence the streamlines coincide with magnetic force lines.
- The invariant subsystem in the singular vortex admits transformations of time translation and two dilatations and allows three symmetry reductions to the ordinary differential equations. Namely, these are stationary, self-similar and logarithmic reductions.
- The detailed description of stationary and self-similar reductions is given. By the integration both reductions are brought to the implicit (not solved with respect to the derivative) ordinary differential equation of the first order. For the partial cases of the solutions the description of the trajectories and magnetic lines is given.

Further, it is necessary to describe the initial velocity and magnetic vector fields. It requires the investigation of solutions of the implicit equation (2.10). The solution of this equation will give the possibility of producing a picture of motion in the whole 3D space.

Acknowledgments

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Appendix. Proof of theorem 4

Properties (a) and (b) are obvious. Let us prove the asymptotics (c). According to the monotonicity of the integral curves all of them start from the discriminant curve. Increasing integral curves, which start in the lower half-plane $\sigma < 0$, go to the upper half-plane $\sigma > 0$ and vice versa. Actually, in the opposite case the integral curve must have a horizontal asymptotic, which is not possible. Let us note the symmetry of the set of integral curves of equation (5.19) under the change $\sigma \to -\sigma$. Therefore, it is enough to observe only the set of increasing solutions in the half-plane $\sigma > 0$. All of them start either from the discriminant curve with horizontal direction or from the axis $\sigma = 0$ with angle less or equal to π. Integral curves cannot intersect the discriminant curve due to the property (b).

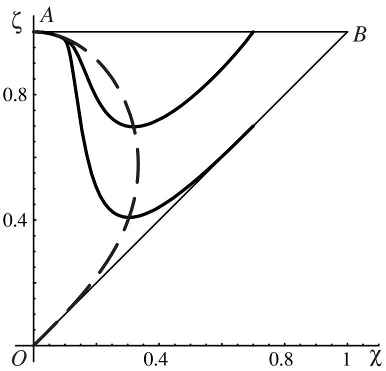

Figure A1. Integral curves of equation (A.2).

Let us perform the change of variables

$$\chi = \xi^{-1}, \qquad \zeta = \xi^{-1}\cosh\sigma. \tag{A.1}$$

Equation (5.19) transforms as follows:

$$\frac{d\zeta}{d\chi} = \frac{\zeta}{\chi} \pm \frac{\sqrt{(1-\zeta^2)(\zeta^2-\chi^2)}}{2\chi^2}. \tag{A.2}$$

According to the propositions we choose the sign '−'. Equation (A.2) is determined in the triangle $T = \{0 < \chi \leqslant 1,\ 0 \leqslant \zeta \leqslant 1, \chi \leqslant \zeta\}$. Let us denote its vertices as $O = (0, 0), A = (0, 1), B = (1, 1)$ (see figure A1). The side AB corresponds to the discriminant curve. The side OA is an image of the infinite point $\xi \to \infty$. The side OB corresponds to the axis $\sigma = 0$. The singular points O and A are the images of infinity with asymptotics $\sigma \to$ const and $\xi^{-1}\cosh\sigma \to 1$ correspondingly. Our goal is to prove that all the integral curves of equation (A.2) start at A. The side OA itself is an integral curve of equation (A.2), therefore no integral curve can intersect it. All integral curves start from the singular point O or A and come to the sides AB or OB.

Let us find the curve of extremum points for the integral curves of (A.2). Setting the right-hand side of (A.2) to zero, after some transformations we have

$$\zeta^4 + (3\chi^2 - 1)\zeta^2 + \chi^2 = 0. \tag{A.3}$$

Curve (A.3) is shown in figure A1 by the dashed line. The integral curves decrease to the left of the curve and increase to the right. Therefore, the integral curve, which starts from the point O, should pass between curve (A.3) and the side OB. Curve (A.3) has the following asymptotics near the points O and A:

$$O: \zeta = \chi + 2\chi^3 + O(\chi^4), \qquad A: \zeta = 1 - 2\chi^2 + O(\chi^4).$$

The integral curve, which starts at the point O, should have the following asymptotic at the point O: $\zeta = \chi + a\chi^3 + O(\chi^4), a \leqslant 2$. One can show that such solutions of (A.2) do not exist. Thus, all integral curves of (A.2) start at the point A. This proves the statement of the theorem. The typical integral curves are shown by the continuous lines in figure A1.

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